On the Exact Solubility in Momentum Space of the Trigonometric Rosen-Morse Potential

C. B. Compean and M. Kirchbach

Instituto de Física,
Universidad Autónoma de San Luis Potosí,
Av. Manuel Nava 6, San Luis Potosí, S.L.P. 78290, México

Abstract: The Schrödinger equation with the trigonometric Rosen-Morse potential in flat three dimensional Euclidean space, E_3, and its exact solutions are shown to be also exactly Fourier transformable to momentum space, though the resulting equation is purely algebraic and can not be cast into the canonical form of an integral Lippmann-Schwinger equation. This is because the cotangent function does not allow for an exact Fourier transform in E_3. In addition we recall, that the above potential can be also viewed as an angular function of the second polar angle parametrizing the three dimensional spherical surface, S^3, of a constant radius, in which case the cotangent function would allow for an exact integral transform to momentum space. On that basis, we obtain a momentum space Lippmann-Schwinger type equation, though the corresponding wavefunctions have to be obtained numerically.

1 Introduction

Constructing the phase spaces of quantum mechanical systems is one of the key issues in their description. The knowledge of the potential and the wave functions in momentum space is indispensable both in relativistic and nonlinear dynamics frameworks. In particular, the trigonometric Rosen-Morse potential (Rosen-Morse I), being of finite range, is becoming of interest in the design of confinement phenomena ranging form electrons in quantum dots [1] to quarks in hadrons [2], a reason for which studying its momentum space representation appears timely.

In a flat three dimensional (3D) Euclidean position space,E_3, the Schrödinger equation,

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) = E\psi(\mathbf{r}),\tag{1}$$

provides a suitable tool for the description of systems interacting via central potentials, many of which result exactly solvable [4]. The momentum space counterpart to eq. (1) is obtained by means of a Fourier transform [5] as

$$\frac{\mathbf{p}^2}{2\mu}\phi(\mathbf{p}) + \int e^{\frac{i}{\hbar}\mathbf{P}\cdot\mathbf{r}}V(\mathbf{r})\psi(\mathbf{r})d^3\mathbf{r} = E\phi(\mathbf{p}),\tag{2}$$

with $\phi(\mathbf{p})$ being the Fourier transform of $\psi(\mathbf{r})$,

$$\phi(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int e^{\frac{i}{\hbar}\mathbf{P}\cdot\mathbf{r}}\psi(\mathbf{r})d^3\mathbf{r},\tag{3}$$

1

and supposed to exist. In cases where the Fourier transform of the potential too exists, i.e.

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{\frac{i}{\hbar}\mathbf{q}\cdot\mathbf{r}} V(\mathbf{r}) d^3\mathbf{r} = V(\mathbf{q}), \tag{4}$$

the momentum space Schrödinger equation can be cast into the canonical form,

$$\frac{\mathbf{p}^2}{2\mu}\phi(\mathbf{p}) + \int V(\mathbf{q})\phi(\mathbf{p}')d^3\mathbf{p}' = E\phi(\mathbf{p}), \quad \mathbf{q} = \mathbf{p} - \mathbf{p}'. \tag{5}$$

The latter equation is frequently referred to as Lippmann-Schwinger equation. From now onwards we introduce $F(\mathbf{p})$ as a special symbol for the integral appearing in eq. (2), i.e. we define

$$F(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} V(\mathbf{r})\psi(\mathbf{r})d^3\mathbf{r}. \tag{6}$$

Provided, the integral in eq. (4) exists, an equivalent representation of eq. (6) is obtained as

$$F(\mathbf{p}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int V(\mathbf{q})\phi(\mathbf{p}')d^3\mathbf{p}'. \tag{7}$$

In case the integrals in eqs.(3) and (6) can be taken in close forms, while the integral in eq. (4) does not exist, eq. (2) is purely algebraical. It is the goal of the present study to show that this precisely is the case of the trigonometrical Rosen-Morse potential (Rosen-Morse I) when considered as a central potential in E_3. The resulting momentum space is then conjugate to the infinite flat position space.

Various phase spaces can be designed in considering the argument of Rosen-Morse I as the arc of S^3, the minimal geodesic distance on a three dimensional space of constant positive curvature. In particular we consider the arc in its projection on the equatorial disk, a 3D space of finite volume, and show that for this very case the momentum space Schrödinger equation is no longer algebraic but takes a form similar to the integral equation (5).

The paper is organized as follows. In the next section we present the cotangent interaction from the perspective of potential theory. Section 3 contains the algebraic momentum space equation. The integral momentum space equation is given in Section 4. The paper closes with brief conclusions.

2   The cotangent interaction from the perspective of potential theory

The trigonometric Rosen-Morse potential (up to additive constants) is given by

$$V(\chi) = -2B\cot\chi + \frac{\hbar^2}{2\mu d^2}l(l+1)\csc^2\chi, \tag{8}$$

where χ is a dimensionless angular variable, μ has the dimensionality of mass, and d is a matching length parameter. It has the virtue to be one among the few potentials that can be introduced along the line of potential theory [6]. According to potential theory, functions which solve the Laplace-Beltrami equation on a manifold of interest, the so called *harmonic* functions, have the remarkable property, if employed as interactions, to preserve the symmetry of the free geodesic motion [6]. It has been observed long ago by Schrödinger in [7] that $\cot\chi$ is a harmonic function on the three-dimensional (3D) surface of constant positive curvature, the hypersphere S^3 embedded in a flat Euclidean space of four dimensions, E_4, where χ can be viewed as the second polar angle. The isometry group of S^3 is $SO(4)$ and the cotangent function, being harmonic there, respects this very symmetry [8]. The latter is best understood by first noting that the Schrödinger equation with the $\csc^2$ potential is closely related to the eigenvalue problem of the 4D angular momentum on S^3. Through the paper we consider ordinary Euclidean flat space, E_3, embedded in a 4D Euclidean space, E_4, and parametrize the 3D spherical surface S^3 as $x_4^2 + \mathbf{r}^2 = R^2$ with $x_4 = R\cos\chi$, and $|\mathbf{r}| = R\sin\chi$. Here, $\chi \in [0,\pi]$ is the second polar angle in E_4, while R

stands for the constant hyper-radius of S^3. The 4D Laplace-Beltrami operator, $\boxed{}$, is proportional to the operator of the squared 4D angular momentum, $\mathcal{K}^2$, for constant radius, and is given by

$$\boxed{} = -\frac{1}{R^2}\mathcal{K}^2. \tag{9}$$

The analogue on the 2D sphere, S^2, of constant radius $|\mathbf{r}| = a$, is the well known relation $\vec{\nabla}^2 = -\frac{1}{a^2}\mathbf{L}^2$. Consequently, the Schrödinger equation for free geodesic motion on S^3 becomes

$$\left[\frac{\hbar^2}{2\mu R^2}\mathcal{K}^2 - \mathcal{E}\right]\psi(\chi) = 0. \tag{10}$$

The $\mathcal{K}^2$ eigenvalue-problem reads [9]

$$\mathcal{K}^2|Klm\rangle = K(K+2)|Klm\rangle, \quad |Klm\rangle \in \left(\frac{K}{2}, \frac{K}{2}\right), \tag{11}$$

and the $|Klm>$-levels belong to irreducible $SO(4)$ representations of the type

$$|Klm\rangle \in \left(\frac{K}{2}, \frac{K}{2}\right). \tag{12}$$

The quantum numbers, K, l, and m define the eigenvalues of the respective four–, three– and two–dimensional angular momentum operators upon the state. These quantum numbers correspond to the

$$\begin{array}{ccccc} SO(4) & \supset & SO(3) & \supset & SO(2), \\ K & & l & & m, \end{array} \tag{13}$$

reduction chain and satisfy the branching rules,

$$K = 0, 1, 2..., \infty, \quad l = 0, 1, 2, ..K, \quad m = -l, ..., +l. \tag{14}$$

Therefore, the spectrum of eq. (10) is given by,

$$\mathcal{E} = \frac{\hbar^2}{2\mu R^2}K(K+2) = \frac{\hbar^2}{2\mu R^2}(K+1)^2 - \frac{\hbar^2}{2\mu R^2}. \tag{15}$$

It is no more but the spectrum of the 4D rigid rotor. In terms of χ eq. (10) takes the explicit form

$$\frac{\hbar^2}{2\mu R^2}\left[\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} - \frac{\mathbf{L}^2(\theta,\varphi)}{\sin^2\chi}\right]\psi(\chi) - \mathcal{E}\psi(\chi) = 0. \tag{16}$$

Multiplying eq. (16) by $(-\sin^2\chi)$ and changing the variable $\psi(\chi)$ to $\sin\chi\mathcal{S}(\chi)$, results in the following Schrödinger equation

$$\left[-\frac{\hbar^2}{2\mu R^2}\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + U_l(\chi)\right]\mathcal{S}(\chi) = \mathcal{E}\mathcal{S}(\chi),$$

$$U_l(\chi) = \frac{\hbar^2}{2\mu R^2}l(l+1)\csc^2\chi, \tag{17}$$

with $U_l(\chi, \kappa)$ now having the meaning of centrifugal barrier on S^3. As a different reading to eqs. (10), and (17) one can say that the $\csc^2$ potential, in representing the centrifugal barrier on the 3D hypersphere, has $SO(4)$ as potential algebra. The solutions to eq. (17) are determined by the Gegenbauer polynomials, $C_n^m(\cos\chi)$, as

$$\mathcal{S}(\chi) = \mathcal{N}_{Kl}\sin^l\chi C_{K-l}^{l+1}(\cos\chi), \tag{18}$$

3

where $\mathcal{N}_{Kl}$ is a normalization constant.

Introducing now the S^3 harmonic function ($-2G/R \cot \chi$) as an interaction into eq. (16) results in:

$$S^3: \quad \frac{\hbar^2}{2\mu R^2}\left[\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} - \frac{\mathbf{L}^2(\theta,\varphi)}{\sin^2\chi}\right]\Psi(\chi) - 2G\frac{\cot\chi}{R}\Psi(\chi) - E\Psi(\chi) \quad = \quad 0. \quad (19)$$

The extension of eq. (17) by same interaction amounts to

$$S^3: \quad \left[-\frac{\hbar^2}{2\mu R^2}\frac{\mathrm{d}^2}{\mathrm{d}\chi^2} + \frac{\hbar^2}{2\mu R^2}l(l+1)\csc^2\chi - 2G\frac{\cot\chi}{R}\right]\psi(\chi) = E\psi(\chi), \quad B = \frac{G}{R}, \quad (20)$$

and describes $SO(4)$ symmetric one-particles motion on S^3 within a cotangent field of a source placed on the "Northern pole" of the hypersphere. This equation is exactly solvable and the solutions are either expressed in terms of Jacobi polynomials of a purely imaginary argument and complex parameters that are conjugate to each-other [10], [11] or, alternatively, in terms of the framework of the real Romanovski polynomials, recently elaborated in [12], [3] and reviewed in [13]. The spectrum of eq. (20) now reads

$$E = \frac{\hbar^2}{2\mu R^2}K(K+2) - \frac{G^2 2\mu}{\hbar^2(K+1)^2}. \quad (21)$$

From the latter formula one reads off that in the limit of an infinite radius, i.e. $\lim R \to \infty$, corresponding to the expansion of the hypersphere towards the E_3 space, in which the contribution of the curved centrifugal barrier tends to zero, the spectrum becomes hydrogen-like [14]. This is the virtue of having introduced the cotangent potential in eq. (20) as inversely proportional to R. Note, however, that a complete E_4 wave equation with an angular potential introduced in this manner wouldn't be separable in polar coordinates.

It is furthermore important to notice that the trigonometric Rosen-Morse potential describes a system put on a finite volume and thereby a confinement phenomenon, as visible through the fact that its spectrum is exclusively discrete. Yet, in the limit of a vanishing curvature, $\lim \kappa \to 0$, with $\kappa = 1/R^2$, the volume increases toward infinity and the space flattens. In the more specific limit of

$$\lim \frac{(K+1)}{R} \xrightarrow{R\to\infty, K\to\infty} k, \quad k = \mathrm{const}: \quad \lim E \to \frac{\hbar^2 k^2}{2\mu}, \quad (22)$$

the energies of the high-lying bound excitations start approaching the energies of the scattering continuum states of the Coulomb potential [14]. In effect, the process of shutting down the curvature of Rosen-Morse I acquires features of a deconfinement transition. This particular property of Rosen-Morse I makes it attractive for the description of confinement phenomena such as quantum dots [1] and quark confinement [2].

Back to the 3D curved manifold under consideration, we recall that the χ angle is measured in terms of the arc, $\widehat{r}$, along the geodesic distance on S^3, as

$$S^3: \quad \chi = \frac{\widehat{r}}{R}. \quad (23)$$

Flat three dimensional position spaces can be approached through various parameterizations of χ. In one of the possibilities, the χ angle can be parametrized in terms of the radial distance, r, from origin within the finite volume 3D flat space of the equatorial disk, $\mathcal{D}_3$, of the hypersphere, according to

$$\mathcal{D}_3: \quad \chi = \sin^{-1}\left(\frac{r}{R}\right), \quad r \in [0, R], \quad \mathbf{r} \in \mathcal{D}_3 \subset S^3. \quad (24)$$

4

This is the parametrization used in the Robertson-Walker metric of a closed universe for which eq. (19) takes the form

$$\mathcal{D}_3: \quad \left[-\frac{\hbar^2}{2\mu R^2} \frac{\sqrt{1-\frac{r^2}{R^2}}}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} \sqrt{1-\frac{r^2}{R^2}} r^2 \frac{\mathrm{d}}{\mathrm{d}r} + \frac{\hbar^2}{2\mu} \frac{l(l+1)}{r^2} - 2G \frac{\sqrt{1-\frac{r^2}{R^2}}}{r} \right] \psi\left(\frac{r}{R}\right) \;=\; E\psi\left(\frac{r}{R}\right),$$

$$\psi\left(\frac{r}{R}\right) \overset{\mathrm{df}}{:=} \Psi\left(\sin^{-1}\left(\frac{r}{R}\right)\right). \tag{25}$$

In the physics of quantum dots mentioned above, another parametrization, $\chi = \tan^{-1} r/R$, has been used [1] which projects the surface of the hypersphere on the infinite 3D flat plane tangential to the North pole. Also in this case, the position space Schrödinger equation contains beyond the standard kinetic piece, several gradient terms needed to recover the finite range confinement, as also happens in eq. (25).

Instead, the parametrization used by Schrödinger [7] himself and subsequently adopted by SUSYQM [4] is,

$$E_3: \quad \chi = \frac{r}{R}\pi = \frac{r}{d}, \quad d = \frac{R}{\pi}, \quad \frac{r}{R} \in [0,1]. \tag{26}$$

In the latter parametrization the $(\cot + \csc^2)$ interaction formally acquires features of a central E_3 flat space potential and the respective Schrödinger equation obtained now from eq. (20) takes the well known standard form of wide spread,

$$E_3: \quad \left[-\frac{\hbar^2}{2\mu d^2} \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{1}{d^2} \frac{\hbar^2}{2\mu} l(l+1) \csc^2\left(\frac{r}{d}\right) - 2B \cot\left(\frac{r}{d}\right) \right] \psi\left(\frac{r}{d}\right) = E\psi\left(\frac{r}{d}\right), \quad B = \frac{G}{d}. \tag{27}$$

It is the goal of the present study to transform above eqs. (27) and (25) to momentum space. We hope that the reader will not be confused by the repeated use of same symbol ψ as a wavefunction but will figure out it significance from the relevant context. Before moving further we wish to emphasize that the variable changes in the master equation (20) of course do not affect physics and the energy spectra of eqs. (20), (27), and (25) stay same.

3 The algebraic momentum space equation

Our first point is that the exact solutions of eq. (27) constructed earlier by us in [12], [3] allow for exact Fourier transforms to momentum space. As a reminder, as well as for the sake of self-sufficiency of the presentation and illustrative purposes, we here bring these solutions for the first three S wave states, displayed in Fig. 1.

Subsequently, the solutions of eq. (27) will be specified by the quantum numbers defined in eqs. (12), (13), and (14) as $\Psi_{Klm}(z)$ with $z = r/d$. The following expressions hold valid:

$$\Psi_{0,0,0}(z) = N_0 \frac{\sin z}{z} e^{-bz}, \quad z = \frac{r}{d}, \quad b = \frac{B2\mu d^2}{\hbar^2}, \tag{28}$$

$$\Psi_{1,0,0}(z) = N_1 \left(\cos z - \frac{b}{2} \sin z \right) \frac{\sin z}{z} e^{-b\frac{z}{2}}, \tag{29}$$

$$\Psi_{2,0,0}(z) = N_2 \left(3\cos^2 z - 2b \sin z \cos z + \left(2\left(\frac{b}{3}\right)^2 - 1 \right) \sin^2 z \right) \frac{\sin z}{z} e^{-b\frac{z}{3}}. \tag{30}$$

The normalization constants is given by

$$N_K = (-1)^K \left(\frac{1}{\pi d^3} \right)^{1/2} \left(\frac{b}{K+1} \left(\left(\frac{b}{K+1} \right)^2 + (K+1)^2 \right) \right)^{1/2}, \tag{31}$$
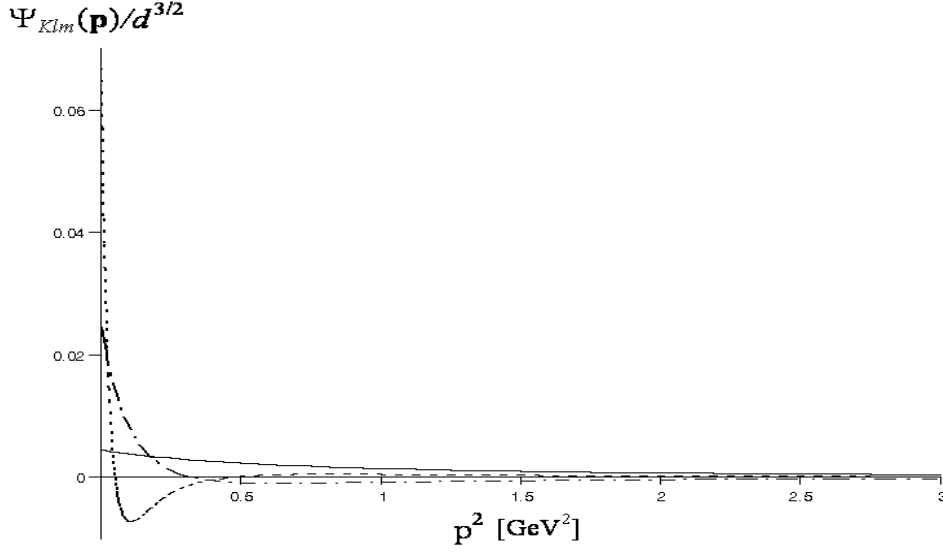
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Figure 1: Momentum dependence of the lowest S wave states in eq. (35). The solid, dashed-dotted, and dotted lines correspond to $K = 0$, $K = 1$, and $K = 2$, respectively.

for $K = 0, 1, 2, ...$ In introducing the short-hand $\bar{p}$ as

$$\bar{p} = \frac{pd}{\hbar}, \quad p = |\mathbf{p}|, \tag{32}$$

the following Fourier transforms of the above wavefunctions are now obtained in close form:

$$\phi_{0,0,0}(\bar{p}) = \left(\frac{2bd}{\hbar}\right)^{\frac{3}{2}} \frac{(b^2+1)^{\frac{1}{2}}}{\pi} \frac{1}{\bar{p}^4 + 2(b^2-1)\bar{p}^2 + 2b^4 + 1}, \tag{33}$$

$$\phi_{1,0,0}(\bar{p}) = -\frac{2^{\frac{1}{2}}}{\pi}\left(\frac{2bd}{\hbar}\right)^{\frac{3}{2}}\left(\left(\frac{b}{2}\right)^2 + 1\right)^{\frac{1}{2}}$$

$$\frac{\bar{p}^2 - \left(\frac{b}{2}\right)^2 - 2}{\bar{p}^6 + \left(3\left(\frac{b}{2}\right)^2 - 8\right)\bar{p}^4 + \left(3\left(\frac{b}{2}\right)^4 + 16\right)\bar{p}^2 + \left(\left(\frac{b}{2}\right)^4 + 8\left(\frac{b}{2}\right)^2 + 16\right)\left(\frac{b}{2}\right)^2}, \tag{34}$$

$$\phi_{2,0,0}(\bar{p}) = \frac{\sqrt{3}}{\pi}\left(\frac{2bd}{\hbar}\right)^{\frac{3}{2}}\left(\left(\frac{b}{3}\right)^2 + 1\right)^{\frac{1}{2}}\left(\bar{p}^4 - \left(10\left(\frac{b}{3}\right)^2 + 22\right)\frac{\bar{p}^2}{3} + \left(\frac{b}{3}\right)^4 + \frac{22}{3}\left(\frac{b}{3}\right)^2 + \frac{19}{3}\right)$$

$$\left[\bar{p}^8 + 4\left(\left(\frac{b}{3}\right)^2 - 5\right)\bar{p}^6 + 2\left(\left(\frac{b}{3}\right)^4 - 10\left(\frac{b}{3}\right)^2 + 59\right)\bar{p}^4 + 4\left(\left(\frac{b}{3}\right)^6 + 5\left(\frac{b}{3}\right)^4 + 23\left(\frac{b}{3}\right)^2 + 45\right)\bar{p}^2\right.$$

$$\left. + \left(\left(\frac{b}{3}\right)^8 + 20\left(\frac{b}{3}\right)^6 + 118\left(\frac{b}{3}\right)^4 + 20\left(\frac{b}{3}\right) + 81\right)\right]^{-1} \tag{35}$$

In employing symbolic programs, the existence of exact Fourier transforms for the other states can be checked. The same applies to the related integrals in eq. (6). As a consequence, the following algebraic equation emerges:
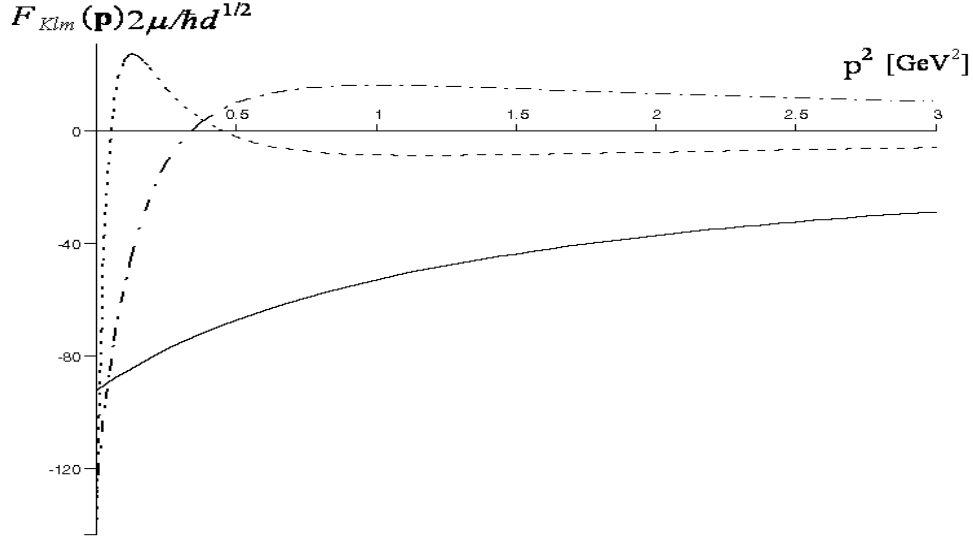
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Figure 2: Momentum dependence of the F_{Klm} quantity in eq. (36) for the lowest S wave states. The solid, dashed-dotted, and dotted lines refer to $K = 0$, $K = 1$, and $K = 2$, respectively.

$$\left(E - \frac{\mathbf{p}^2}{2\mu} \right) \phi_{Klm}(\mathbf{p}) \;\; = \;\; F_{Klm}(\mathbf{p}),$$

$$F_{Klm}(\mathbf{p}) \;\; = \;\; \frac{1}{(2\pi)^{\frac{3}{2}}} \int e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}} \left(-\frac{2G}{d} \right) \cot\left(\frac{r}{d} \right) \Psi_{Klm}\left(\frac{r}{d} \right) Y_l^m(\theta,\varphi) \mathrm{d}^3\mathbf{r}. \tag{36}$$

Below we bring the explicit expressions for F_{Klm} for the two lowest S states, displayed in Fig. 2, for illustrative purposes,

$$F_{0,0,0}(\bar{p}) \;\; = \;\; \frac{1}{\pi} \left(\frac{2bd}{\hbar} \right)^{\frac{3}{2}} (b^2 + 1)^{\frac{1}{2}} \frac{1 - b^2 - \bar{p}^2}{\bar{p}^4 + 2(b^2 - 1)\bar{p}^2 + 2b^4 + 1} \tag{37}$$

$$F_{1,0,0}(\bar{p}) \;\; = \;\; -\frac{2^{\frac{1}{2}}}{\pi} \left(\frac{2bd}{\hbar} \right)^{\frac{3}{2}} \left(\left(\frac{b}{2} \right)^2 + 1 \right)^{\frac{1}{2}}$$

$$\frac{\left(\frac{b}{2} \right)^4 - 2\left(\frac{b}{2} \right)^2 - 8 + 6\bar{p}^2 - \bar{p}^4}{\bar{p}^6 + \left(3\left(\frac{b}{2} \right)^2 - 8 \right)\bar{p}^4 + \left(3\left(\frac{b}{2} \right)^4 + 16 \right)\bar{p}^2 + \left(\left(\frac{b}{2} \right)^4 + 8\left(\frac{b}{2} \right)^2 + 16 \right)\left(\frac{b}{2} \right)^2} . \tag{38}$$

4 The integral momentum space equation

Equation (36) from the previous section is quite transparent indeed, however it doesn't make the momentum space potential explicit. This may be perceived as a shortcoming by T-matrix based spectroscopic approaches [15]. Below we shall show that such a shortcoming is not suffered by the Fourier transformed eq. (25), obtained from the master equation (20) via the change of variable according to eq. (24). In introducing the short hand,

$$x = \frac{r}{R}, \quad r = |\mathbf{r}|, \tag{39}$$

7

eq. (25) equivalently rewrites to

$$-\frac{\sqrt{1-x^2}}{x^2}\frac{d}{dx}x^2\sqrt{1-x^2}\frac{d}{dx}\psi(x) + \frac{l(l+1)}{x^2}\psi(x) - 2b\frac{\sqrt{1-x^2}}{x}\psi(x) - \epsilon\psi(x) = 0. \qquad (40)$$

where we in addition have divided by $\hbar^2/2\mu R^2$ and switched to dimensionless constants by introducing the two new notations,

$$b = \frac{G}{R}\left(\frac{\hbar^2}{2\mu R^2}\right)^{-1}, \quad \epsilon = E\left(\frac{\hbar^2}{2\mu R^2}\right)^{-1}. \qquad (41)$$

After some algebraic manipulations equation (40) takes the form

$$-\left(\nabla^2 - \frac{1}{x^2}\frac{d}{dx}x^2\frac{d}{dx}x^2 + 3x\frac{d}{dx} + 6\right)\psi(x) - 2b\frac{\sqrt{1-x^2}}{x}\psi(x) - \epsilon\psi(x) = 0. \qquad (42)$$

This is the equation which we shall Fourier transform to momentum space. We first obtain the integral transform of the interaction piece of the Hamiltonian as

$$\frac{-2b}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\sqrt{1-x^2}}{x}\psi(x) = -\frac{1}{2(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}'\int d^3\mathbf{k}''\Pi(|\mathbf{q}|)\frac{(J_0(\mathbf{k}'') - 2J_1(\mathbf{k}''))}{(\mathbf{k}'')^2}\phi(\mathbf{k}'),$$

$$\mathbf{q} = \mathbf{k} - \mathbf{k}' - \mathbf{k}''. \qquad (43)$$

Here, use has been made of

$$\sqrt{1-x^2} = -\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{k}''\, e^{-i\mathbf{k}''\cdot\mathbf{x}}\sqrt{\frac{\pi}{2}}\frac{(J_0(\mathbf{k}'') - 2J_1(\mathbf{k}''))}{(\mathbf{k}'')^2}, \quad x = |\mathbf{x}| \leq 1, \qquad (44)$$

and

$$\frac{1}{x} = \frac{1}{(2\pi)^2}\int d^3\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{\sin^2\frac{|\mathbf{q}|}{2}}{\left(\frac{\mathbf{q}}{2}\right)^2} = \frac{1}{(2\pi)^2}\int d^3\mathbf{q}\, e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{1}{(-b)}\Pi(|\mathbf{q}|), \quad x = |\mathbf{x}| \leq 1. \qquad (45)$$

In the latter equation we worked in the momentum-space potential, $\Pi(|\mathbf{q}|)$, obtained earlier by us in ref. [16] as

$$\Pi(|\mathbf{q}|) = (-b)\frac{\sin^2\frac{|\mathbf{q}|}{2}}{\left(\frac{\mathbf{q}}{2}\right)^2}, \qquad (46)$$

the E_4 Fourier transform of $(-2b\cot\chi)$ for the elastic scattering case, $q_4 = 0$, which we calculated using the S^3 integration volume according to

$$4\pi\Pi(|\mathbf{q}|) = -2b\int_0^\infty d|x||x|^3\delta(|x| - 1)\int_0^{2\pi}d\varphi\int_0^\pi d\theta\sin\theta$$

$$\int_{0/\frac{\pi}{2}}^{\frac{\pi}{2}/\pi}d\chi\sin^2\chi e^{i|\mathbf{q}|\frac{\sin\chi}{\sqrt{\kappa}}|\cos\theta}\cot\chi. \qquad (47)$$

We used the $\delta(|x| - 1)$ function in order to restrict E_4 to S^3. In the context of the present study and the dimensionless variable in equation (39), S^3 presents itself as the unit hypersphere.

Finally, transforming the pieces of the kinetic part in eq. (42) amounts to

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}x\frac{d\psi(x)}{dx} = -3\phi(\mathbf{k}) - \mathbf{k}\cdot\nabla_\mathbf{k}\phi(\mathbf{k}), \qquad (48)$$

8

and

$$\frac{1}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}} x\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}x\psi(x) \quad = \quad \mathbf{k}\cdot\nabla_{\mathbf{k}}\mathbf{k}\cdot\nabla_{\mathbf{k}}\phi(\mathbf{k}) + 6\,\mathbf{k}\cdot\nabla_{\mathbf{k}}\phi(\mathbf{k}) + 9\phi(\mathbf{k}), \tag{49}$$

respectively.

Putting it all together the integral momentum space equation becomes

$$\left(\mathbf{k}^2 + \mathbf{k}\cdot\nabla_{\mathbf{k}}\mathbf{k}\cdot\nabla_{\mathbf{k}} + 4\,\mathbf{k}\cdot\nabla_{\mathbf{k}} + 3\right)\phi(\mathbf{k}) - \frac{1}{2(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3\mathbf{k}'\phi(\mathbf{k}')\int \mathrm{d}^3\mathbf{k}''\Pi(|\mathbf{q}|)\frac{J_0(\mathbf{k}'') - 2J_1(\mathbf{k}'')}{(\mathbf{k}'')^2} \quad = \quad \epsilon\phi(\mathbf{k}). \tag{50}$$

This equation is satisfied by the numerical Fourier transforms of the wavefunctions from position space.

5   Conclusions

We recalled that the most natural view on the trigonometric Rosen-Morse potential appears within the context of a particle moving in the cotangent field of a source placed at one of the poles of the three-dimensional spherical surface, S^3, of a constant radius, embedded within a four-dimensional Euclidean space, E_4. Within this context the trigonometric Rosen-Morse potential has been considered as an angular function of the arc of the minimal geodesic distance on S^3 and the related Schrödinger equation has been presented in eq. (20). This general view on Rosen-Morse I has the virtue of making its $SO(4)$ symmetry obvious. Indeed, within this context, the $\csc^2\chi$ part plays the part of a centrifugal barrier on S^3 whose isomerty group is $SO(4)$, a symmetry respected by the $\cot\chi$ piece, known to be a harmonic function on this manifold. We then considered two different parameterizations of the arc in eq. (23) and presented them in eqs. (24), and (26). We wrote down the related position space Schrödinger equations in eqs. (25), and (27), respectively. We noted that equation (27) coincides with the Schrödinger equation with Rosen-Morse I manged by SUSYQM, where the potential is treated as central in ordinary three-dimensional flat Euclidean space, E_3. We subjected eqs. (25) and (27) to ordinary 3D Fourier transformations to momentum space. We showed that in the first case one finds a genuine integral, in the second an exactly solvable algebraic equation.

We expect our findings to be of use in momentum space many-body frameworks based on the T-matrix.

References

[1] V. V. Gritsev, Yu. A. Kurochkin, *Model of excitations in quantum dots based on quantum mechanics in spaces of constant curvature,* Phys. Rev. B **64**, 035308-1-035308-9 (2001).

[2] M. Kirchbach, C. B. Compean, *Conformal symmetry and light flavor baryon spectra,* Phys. Rev. D **82**, 034008 (2010).

[3] C. B. Compean, M. Kirchbach, *Trigonometric quark confinement potential of QCD traits,* Eur. Phys. J. A **33**, 1-4 (2007).

[4] F. Cooper, A. Khare, U. Sukhatme, *Supersymmetry in Quantum Mechanics*, Phys. Rept. **251**, 267-385 (1995)

[5] Alfred Tang, *MKN Theory of Bound States,* hep-ph/0103035v4

[6] Sheldon Axler, Paul Bourdon, Wade Ramey, *Harmonic Function Theory,* (Springer Verlag N.Y.Inc, 2001); S. A. Gabov, *The angular potential and some of its applications,* Math. USSR Sbornik **32**, 423-436 (1977): http//iopscience.iop.org/0025-5734/32/4/A03.

[7] E. Schrödinger, *A method of determining quantum mechanical eigenvalues and eigenfunction,* Proc. Roy. Irish Acad. A **46**, 9-16 (1940).

[8] P. W. Higgs, *Dynamical symmetries in a spherical geometry,* J. Phys A:Math.Gen. **12**, 309-323 (1979).

[9] Y. S. Kim, M. E. Noz, *Theory and application of the Poincaré group* (D. Reidel, Dordrecht, 1986).

[10] G. Levai, *A search for shape invariant solvable potentials,* J. Phys. A:Math. Gen. **22**, 689 (1989).

[11] R. De, R. Dutt, U. Sukhatme, *Mapping of shape invariant potentials under point canonical transformations,* J. Phys. A:Math.Gen. **25**, L843-L850 (1992).

[12] C. B. Compean, M. Kirchbach, *The trigonometric Rosen-Morse potential in the supersymmetric quantum mechanics and its exact solutions,* J. Phys. A:Math.Gen. **39**, 547-557 (2006).

[13] A. Raposo, H.-J. Weber, D. E. Alvarez-Castillo, M. Kirchbach, *Romanovski polynomials in selected physics problems,* C. Eur. J. Phys. **5**, 253-284 (2007).

[14] A. O. Barut, Raj Wilson, *On the dynamical group of the Kepler problem in a curved space of constant curvature,* Phys. Lett. A **110**, 351-354 (1985).

[15] E. Oset, A. Ramos, *Dynamically generated resonances from the vector octet-baryon interaction,* Eur. Phys. J. A **44**, 445-454 (2010).

[16] C. B. Compean, M. Kirchbach, *Momentum space trigonometric Rosen Morse potential,* J. Phys. A:Math.Theor. **42**, 365301 (2009).